BUSEMANN POINTS OF ARTIN GROUPS OF DIHEDRAL TYPE

CORMAC WALSH

Abstract. We study the horofunction boundary of an Artin group of dihedral type with its word metric coming from either the usual Artin generators or the dual generators. In both cases, we determine the horoboundary and say which points are Busemann points, that is the limits of geodesic rays. In the case of the dual generators, it turns out that all boundary points are Busemann points, but this is not true for the Artin generators. We also characterise the geodesics with respect to the dual generators, which allows us to calculate the associated geodesic growth series.

1. Introduction

Consider the following metric space boundary, defined first by Gromov [11]. One assigns to each point z in the metric space (X, d) the function $\psi_z : X \to \mathbb{R}$,

$$\psi_z(x) := d(x, z) - d(b, z),$$

where b is some basepoint. If X is proper and complete, then the map $\psi : X \to C(X)$, $z \mapsto \psi_z$ defines an embedding of X into $C(X)$, the space of continuous real-valued functions on X endowed with the topology of uniform convergence on compacts. The horofunction boundary is defined to be $X(\infty) := \mathrm{cl}\{\psi_z \mid z \in X\} \backslash \{\psi_z \mid z \in X\}$, and its elements are called horofunctions.

This boundary is not the same as the better known Gromov boundary of a δ-hyperbolic space. For these spaces, it has been shown [7, 16, 15] that the horoboundary is finer than the Gromov boundary in the sense that there exists an equivariant continuous surjection from the former to the latter.

An interesting class of metric spaces are the Cayley graphs of finitely generated groups with their word metric. Here one may hope to have a combinatorial description of the horoboundary. Rieffel [13] has investigated the horoboundary in this setting. A length function on a discrete group naturally gives rise to a metric on the state space of the reduced group C*-algebra [6] and, in the case of $\mathbb{Z}^d$ with a word metric coming from a finite set of generators, Rieffel used the horoboundary to determine certain properties of this metric, in particular, whether it is compatible with the weak* topology on the state space.

This motivates the study of the horoboundary of other finitely generated groups. In this paper, we investigate the horofunction boundary of the Artin groups of dihedral type. Let $\mathrm{prod}(s, t; n) := stst\cdots$, with n factors in the product. The

Date: February 1, 2008.

2000 *Mathematics Subject Classification.* Primary 20F36; 20F65.

Key words and phrases. Artin groups, braid groups, Garside groups, geodesics, growth series, horoball, max-plus algebra, metric boundary, Busemann function.

Artin groups of dihedral type have the following presentation:

$$A_k = \langle a, b \mid \operatorname{prod}(a, b; k) = \operatorname{prod}(b, a; k)\rangle, \qquad \text{with } k \geq 3.$$

Observe that A_3 is the braid group on three strands. The generators traditionally considered are the Artin generators $S := \{a, b, a^{-1}, b^{-1}\}$.

In what follows, we will have need of the Garside normal form for elements of A_k. The element $\Delta := \operatorname{prod}(a, b; k) = \operatorname{prod}(b, a; k)$ is called the Garside element. Let

$$M^+ := \{a, b, ab, ba, \ldots, \operatorname{prod}(a, b; k-1), \operatorname{prod}(b, a; k-1)\}.$$

It can be shown [9] that $w \in A_k$ can be written

$$w = w_1 \cdots w_n \Delta^r$$

for some $r \in \mathbb{Z}$ and $w_1, \ldots, w_n \in M^+$. This decomposition is unique if n is required to be minimal. We call it the right normal form of w. The factors $w_1, \ldots, w_n$ are called the canonical factors of w.

One can also write w in left normal form: $w = \Delta^r w'_1 \cdots w'_n$, with $r \in \mathbb{Z}$ and $w'_1, \ldots, w'_n \in M^+$.

To calculate the horoboundary, we will need a formula for the word length metric. An algorithm was given in [3] for finding a geodesic word representing any given element of A_3. In [14], there is a criterion for when a word is a geodesic in A_3. Both these results were generalised in [12] to arbitrary $k \geq 3$. It was shown that a freely reduced word u is a geodesic with respect to the Artin generators if and only if

$$\operatorname{Pos}(u) + \operatorname{Neg}(u) \leq k. \tag{1}$$

Here $\operatorname{Pos}(u)$ is the length of the longest possible element of $M^+ \cup \{\Delta\}$ obtainable by multiplying together consecutive letters of u. The length of an element $\operatorname{prod}(a, b; n)$ or $\operatorname{prod}(b, a; n)$ of $M^+ \cup \{\Delta\}$ is defined to be n. Likewise, $\operatorname{Neg}(u)$ is the length of the longest possible element of $M^- \cup \{\Delta^{-1}\}$ obtainable in the same way, where $M^- := (M^+)^{-1}$.

We use the algorithm in [12] to find a simple formula for the word length metric.

Proposition. *Let $x = \Delta^r z_1 \cdots z_m$ be an element of A_k written in left normal form. Let $(p_0, \ldots, p_{k-1}) \in \mathbb{N}^k$ be such that $p_0 := r$ and, for each $i \in \{1, \ldots, k-1\}$, $p_i - p_{i-1} = m_{k-i}$, where m_i is the number of canonical factors of x of length i. Then the distance from the identity e to x in the Artin-generator word-length metric is*

$$d(e, x) = \sum_{i=0}^{k-1} |p_i|.$$

Since d is invariant under left multiplication, that is, $d(y, x) = d(e, y^{-1}x)$, we can use this formula to calculate the distance between any pair of elements y and x of A_k. With this knowledge we can find the following description of the horofunction compactification.

Let Z be the set of possibly infinite words of positive generators having no product of consecutive letters equal to Δ. We can write each element z of Z as a concatenation of substrings in such a way that the products of the letters in every substring equals an element of M^+ and the combined product of letters in each consecutive pair of substrings is not in M^+. Because z does not contain Δ, this

decomposition is unique. Let $m_i(z)$ denote the number of substrings of length i. Note that if z is an infinite word, then this number will be infinite for some i.

Let Ω' denote the set of (p, z) in $(\mathbb{Z} \cup \{-\infty, +\infty\})^k \times Z$ satisfying the following:

- $p_i - p_{i-1} \geq m_{k-i}(z)$ for all $i \in \{1, \ldots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$;
- if z is finite, then $p_i - p_{i-1} = m_{k-i}(z)$ for all $i \in \{1, \ldots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$.

We take the product topology on Ω'.

We now define Ω to be the quotient topological space of Ω' where the elements of $(+\infty, \ldots, +\infty) \times Z$ are considered equivalent and so also are those in $(-\infty, \ldots, -\infty) \times Z$. We denote these two equivalence classes by $+\hat{\infty}$ and $-\hat{\infty}$, respectively.

We let $\mathcal{M}$ denote the horofunction compactification of A_k with the Artin-generator word metric. The basepoint is taken to be the identity.

Theorem. *The sets Ω and $\mathcal{M}$ are homeomorphic.*

Let Z_0 be the set of elements of Z that are finite words. Let Ω_0 denote the set of (p, z) in $\mathbb{Z}^k \times Z_0$ such that $p_i - p_{i-1} = m_{k-i}(z)$ for all $i \in \{1, \ldots, k-1\}$. We will show that the elements of Ω_0 are exactly the elements of Ω corresponding to functions of the form $d(\cdot, z) - d(e, z)$ in $\mathcal{M}$.

Of particular interest are those horofunctions that are the limits of almost-geodesics; see [1] and [13] for two related definitions of this concept. Rieffel calls the limits of such paths Busemann points. In the present context, since the metric takes only integer values, the Busemann points are exactly the limits of geodesics (see [17]). Develin [8], investigated the horoboundary of finitely generated abelian groups with their word metrics and showed that all their horofunctions are Busemann. Webster and Winchester [17] gave a necessary and sufficient condition for all horofunctions of a finitely generated group to be Busemann.

We prove the following characterisation of the Busemann points of A_k.

Theorem. *A function in $\mathcal{M}$ is a Busemann point if and only if the corresponding element (p, z) of Ω is in $\Omega \backslash \Omega_0$ and satisfies the following: $p_i - p_{i-1} = m_{k-i}(z)$ for every $i \in \{1, \ldots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$.*

The group A_k also has a dual presentation:

$$A_k = \langle \sigma_1, \ldots, \sigma_k \mid \sigma_1\sigma_2 = \sigma_2\sigma_3 = \cdots = \sigma_k\sigma_1 \rangle, \qquad \text{with } k \geq 3.$$

The set of dual generators is $\tilde{S} := \{\sigma_1, \ldots, \sigma_k, \sigma_1^{-1}, \ldots, \sigma_k^{-1}\}$. These are related to the Artin generators in the following way: $\sigma_1 = a$, $\sigma_2 = b$, and

$$\sigma_j = \begin{cases} \operatorname{prod}(b^{-1}, a^{-1}; j-2) \operatorname{prod}(a, b; j-1), & \text{if } j \text{ is odd}, \\ \operatorname{prod}(b^{-1}, a^{-1}; j-2) \operatorname{prod}(b, a; j-1), & \text{if } j \text{ is even}, \end{cases}$$

for $j \in \{3, \ldots, k\}$. The existence of a dual presentation holds more generally for all Artin groups of finite type [4].

There are also Garside normal forms related to the dual presentation. Here the Garside element is $\delta := \sigma_1\sigma_2 = \cdots = \sigma_k\sigma_1$.

Again, we find a formula for the word length metric.

Proposition. *Let $w = \delta^r w_1 w_2 \cdots w_s$ be written in left normal form. Then the distance between the identity and w with respect to the dual generators is given by $\tilde{d}(e,w) = |r| + |r+s|$.*

Using this formula, we again determine the horoboundary. This time however, there are no non-Busemann points.

Theorem. *In the horoboundary of A_k with the dual-generator word metric, all horofunctions are Busemann points.*

In general, one would expect the properties of the horofunction boundary of a group with its word length metric to depend strongly on the generating set. It would be interesting to know for which groups and for which properties there is not this dependence. As already mentioned, all boundary points of abelian groups are Busemann no matter what the generating set [8]. On the other hand, the above results show that for Artin groups of dihedral type the existence of non-Busemann points depends on the generating set.

We use our formula to establish a criterion for a word to be a geodesic with respect to the dual generators. For every word y with letters in $\tilde{S}$, let $\widetilde{\mathrm{Pos}}(y)$ be the longest element of $\{\sigma_1, \ldots, \sigma_k, \delta\}$ obtainable by multiplying together consecutive letters of y. The generators $\sigma_1, \ldots, \sigma_k$ are considered to each have length 1 whereas δ is considered to have length 2. Similarly, $\widetilde{\mathrm{Neg}}(y)$ is defined to be the longest element of $\{\sigma_1^{-1}, \ldots, \sigma_k^{-1}, \delta^{-1}\}$ obtainable in the same way.

Proposition. *Let y be a freely reduced word of dual generators. Then y is a geodesic if and only if $\widetilde{\mathrm{Pos}}(y) + \widetilde{\mathrm{Neg}}(y) \le 2$.*

The geodesic growth series of a finitely generated group G with respect to a generating set S is

$$\mathcal{G}_{(G,S)}(x) := \sum_{n=0}^{\infty} a_n x^n,$$

where a_n is the number of words of length n that are geodesic with respect to S.

It is obvious from the characterisation of geodesics given above that the set of geodesic words with respect to the dual set of generators $\tilde{S}$ is a regular language. It follows that the geodesic growth series is rational [9], that is, can be expressed as the quotient of two integer-coefficient polynomials in the ring of formal power series $\mathbb{Z}[[x]]$. We calculate this growth series explicitly.

Theorem. *The geodesic growth series of A_k with the dual generators is*

$$\mathcal{G}(x) = \frac{1 + (3-2k)x + (2+k^2-3k)x^2 - 2k(k-1)x^3}{(1-kx)(1-2(k-1)x)(1-(k-1)x)}.$$

The geodesic growth series has previously been determined for A_k with other generating sets. Charney and Meier [5] calculate it for the generating sets $\{\sigma_1^{\pm}, \ldots, \sigma_k^{\pm}, \delta^{\pm}\}$ and $M^+ \cup M^- \cup \{\Delta^{\pm}\}$. Sabalka [14] calculates it for the 3–strand braid group A_3 with the Artin generators, a result which was generalised by Mairesse and Mathéus [12] to A_k; $k \ge 3$, again with the Artin generators.

2. Artin generators

Proposition 2.1. *Let $x = \Delta^r z_1 \cdots z_m$ be an element of A_k written in left normal form. Let $(p_0, \ldots, p_{k-1}) \in \mathbb{Z}^k$ be such that $p_0 := r$ and, for each $i \in \{1, \ldots, k-1\}$,*

$p_i - p_{i-1} = m_{k-i}$, where m_i is the number of canonical factors of x of length i. Then the distance from the identity e to x in the Artin-generator word-length metric is

$$d(e,x) = \sum_{i=0}^{k-1} |p_i|.$$

Proof. In [12], there is an algorithm for finding a geodesic representative of an element x of A_k given its normal form. This algorithm consists, in the case when $r < 0$, of shifting each instance of Δ^{-1} across and combining it with one of the canonical factors of longest length. This procedure is continued until all the Δ^{-1}s have been moved across or there are no more canonical factors with which to multiply. The resulting word is shown to be a geodesic representative of x.

If $r \geq 0$, then the algorithm leaves the normal form unchanged, and so

$$d(e,x) = \sum_{i=1}^{k-1} i m_i + kr = \sum_{i=0}^{k-1} p_i,$$

which proves the result in this case since here all the p_i are non-negative.

On the other hand, if $-r \geq \sum_{i=1}^{k-1} m_i$, then all the canonical factors are changed: each factor of length $i \in \{1, \dots, k-1\}$ is replaced by a word of length $k - i$. Therefore

$$\begin{aligned} d(e,x) &= \sum_{i=1}^{k-1} (k-i) m_i + k\Big(-r - \sum_{i=1}^{k-1} m_i\Big) \\ &= -kr - \sum_{i=1}^{k-1} i m_i \\ &= -\sum_{i=0}^{k-1} p_i. \end{aligned}$$

But in this case all the p_i are non-positive and we conclude that the result holds here also.

The final case to consider is when $0 < -r < \sum_{i=1}^{k-1} m_i$. In this case, there is some $j \in \mathbb{N}$ such that all factors of length greater than j are changed, all factors of length less than j are unchanged, and possibly some factors of length j are changed. So we have

$$\begin{aligned} d(e,x) &= \sum_{i=1}^{j-1} i m_i + j\Big(\sum_{i=j}^{k-1} m_i + r\Big) + (k-j)\Big(-r - \sum_{i=j+1}^{k-1} m_i\Big) + \sum_{i=j+1}^{k-1} (k-i) m_i \\ &= p_{k-1} + \dots + p_{k-j} - p_{k-j-1} - \dots - p_0. \end{aligned}$$

Because of the choice of j, we have $\sum_{i=j}^{k-1} m_i \geq -r \geq \sum_{i=j+1}^{k-1} m_i$, and so p_i is non-negative for $i \geq k-j$ and non-positive for $i < k-j$. Therefore the result holds in this case also. □

Motivated by this we define the following map. Let z be an element of A_k. For each $i \in \{1, \dots, k\}$, let m_i be the number of canonical factors of length i when z is written in left normal form. We define $\pi : A_k \to \mathbb{Z}^k$ by

$$\pi(z) := (m_k, m_k + m_{k-1}, \dots, m_k + \dots + m_1).$$

Let w and z be two elements of A_k. We define

$$\phi(w,z) := \pi(w^{-1}z) - \pi(z).$$

For $w \in A_k$, denote by $\tau(w)$ the conjugate of w by Δ, that is

$$\tau(w) := \Delta^{-1}w\Delta = \Delta w \Delta^{-1}.$$

Lemma 2.2. *Let $w \in A_k$ and let $z_1 z_2 \cdots$ be an infinite word of positive generators such that no product of consecutive letters equals Δ. Then $\phi(w, z_1 \cdots z_n)$ converges as n tends to infinity.*

Proof. To write $w^{-1}z_1 \cdots z_n$ in left normal form, we first write w^{-1} in left normal form and then repeatedly take the factors Δ formed by the joining of w^{-1} and $z_1 \cdots z_n$ out to the left. We obtain something of the form $\Delta^{r+s}\tau^r(w')z'$, where r is the number of Δs moved, w' is a left divisor of w^{-1} and z' is a right divisor of $z_1 \cdots z_n$. One or both of w' and z' may be the identity. Since w is of finite length, as n is increased z' must eventually be different from the identity, and from then on z' will grow in the same way as $z_1 \cdots z_n$. When z' has grown sufficiently that it contains one of the canonical factors of $z_1 \cdots z_n$, subsequent increases in n will have exactly the same effect on $\pi(w^{-1}z_1 \cdots z_n)$ as on $\pi(z_1 \cdots z_n)$. Therefore $\phi(w, z_1 \cdots z_n)$ is eventually constant. □

Recall that Z is the set of possibly infinite words of positive generators having no product of consecutive letters equal to Δ. The previous lemma allows us to define $\phi(w,z)$ for $w \in A_k$ and $z = z_1 z_2 \cdots$ an infinite element of Z to be the limit of $\phi(w, z_1 \cdots z_n)$ as n tends to infinity.

For each $(p, z) \in \Omega'$, define

$$\psi_{p,z} : A_k \to \mathbb{Z}, \quad w \mapsto \sum_{i=0}^{k-1} |p_i + \phi_i(w,z)| - \sum_{i=0}^{k-1} |p_i|. \tag{2}$$

Note that this formula sometimes requires us to add or subtract infinities. The convention we shall use will be to separately keep track of the infinite and finite parts. Thus $(a\infty + b) + (c\infty + d) = (a + c)\infty + (b + d)$. Obviously, for a and b finite, $|a\infty + b|$ is equal to $a\infty + b$ if $a > 0$, and is equal to $-a\infty - b$ if $a < 0$. We see that $\psi_{p,z}$ is always finite because the infinities in the first term always cancel those in the second.

The following lemma will be needed to show that ψ is constant on the equivalence classes $-\hat{\infty}$ and $+\hat{\infty}$.

Lemma 2.3. *For all w and z in A_k,*

$$\sum_{i=0}^{k-1} \phi_i(w,z) = \sum_{i=0}^{k-1} \pi_i(w^{-1}).$$

Proof. Let $y \in A_k$. Write $y = y_1 \cdots y_s \Delta^r$ in right normal form and let m_i be the number of canonical factors of length i for each $i \in \{1, \ldots, k\}$, so that $m_k = r$. Consider the effect of left multiplying y by a positive generator g. Either g combines with y_1 to form a longer factor, in which case m_i decreases by one and m_{i+1} increases by one, where i is the length of y_1, or a new factor is created, in which case m_1

increases by one. In either case, $\sum_{i=0}^{k-1}(\pi_i(gy) - \pi_i(y)) = 1$. We conclude that

$$\sum_{i=0}^{k-1} \phi_i(g^{-1}, y) = 1, \qquad \text{for all } y \in A_k \text{ and } g \in \{a, b\}. \tag{3}$$

Similar reasoning shows that

$$\sum_{i=0}^{k-1} \phi_i(\Delta, y) = -k, \qquad \text{for all } y \in A_k. \tag{4}$$

Any $w \in A_k$ may be written as a product $w_1 \cdots w_l$ of negative generators and copies of Δ. Observe that

$$\phi(w, z) = \phi(w_1, z) + \phi(w_2, w_1^{-1}z) + \cdots + \phi(w_l, w_{l-1}^{-1} \cdots w_1^{-1}z).$$

Applying (3) and (4), we see that $\sum_{i=0}^{k-1} \phi_i(w, z)$ is independent of z. Therefore

$$\sum_{i=0}^{k-1} \phi_i(w, z) = \sum_{i=0}^{k-1} \phi_i(w, e) = \sum_{i=0}^{k-1} \pi_i(w^{-1}).$$

□

So we see that if p is identically $-\infty$, then

$$\psi_{p,z}(w) = -\sum_{i=0}^{k-1} \pi_i(w^{-1})$$

is independent of z. Likewise, if p is identically $+\infty$, then

$$\psi_{p,z}(w) = \sum_{i=0}^{k-1} \pi_i(w^{-1}).$$

We may therefore consider the map ψ to be defined on Ω.

Define $\mathcal{D} := \{d(\cdot, x) - d(e, x) \mid x \in A_k\}$.

Lemma 2.4. *Restricted to Ω_0, the map ψ is a bijection between Ω_0 and $\mathcal{D}$.*

Proof. Let $(p, z) \in \Omega_0$. Observe that $p_i = \pi_i(z) + p_0 = \pi_i(z\Delta^{p_0})$ for all $0 \le i \le k-1$. For each $w \in A_k$,

$$\begin{aligned}
\psi_{p,z}(w) &= \sum_{i=0}^{k-1} |p_i + \pi_i(w^{-1}z) - \pi_i(z)| - \sum_{i=0}^{k-1} |p_i| \\
&= \sum_{i=0}^{k-1} |\pi_i(w^{-1}z\Delta^{p_0})| - \sum_{i=0}^{k-1} |\pi_i(z\Delta^{p_0})| \\
&= d(w, z\Delta^{p_0}) - d(e, z\Delta^{p_0}).
\end{aligned}$$

The result now follows from the fact that every element of A_k can be written in a unique way as $z\Delta^{p_0}$ with $z \in Z_0$ and $p_0 \in \mathbb{N}$ and that the $p_i; 1 \le i \le k-1$ are determined by z and p_0 for each (p, z) in Ω_0. □

Lemma 2.5. *The set Ω_0 is dense in Ω.*

Proof. Clearly, $-\hat{\infty}$ and $+\hat{\infty}$ are in the closure of Ω_0 since they are the limits, respectively, of $(-n, \dots, -n, e)$ and $(n, \dots, n, e)$, where e denotes the empty word.

Let $(p, z) \in \Omega \backslash \{-\hat{\infty}, +\hat{\infty}\}$ and fix $n \in \mathbb{N}$. Let x_n be the product of the first n canonical factors of z. Define $b_k := \max(\min(p_0, n), -n)$ and $b_{k-i} := \min(p_i - p_{i-1}, n)$ for each $i \in \{1, \dots, k-1\}$. Let m_i denote the number of canonical factors of length i in x_n.

For each $i \in \{1, \dots, k-1\}$, we have that m_i is no greater than the number of canonical factors of length i in z, which is no greater than $p_{k-i} - p_{k-i-1}$. We also have $m_i \leq n$. Therefore $m_i \leq b_i$ for all $i \in \{1, \dots, k-1\}$. So we may multiply x_n on the right by canonical factors to obtain a word y_n of positive generators such that no product of consecutive letters equals Δ and such that, for each $i \in \{1, \dots, k-1\}$, there are exactly b_i factors of length i.

So $(q_n, y_n) := (b_k, b_k + b_{k-1}, \dots, b_k + \dots + b_1, y_n)$ is in Ω_0.

As n tends to infinity, b_k converges to p_0 and b_i converges to $p_{k-i} - p_{k-i-1}$ for $1 \leq i \leq k-1$. So $\sum_{i=0}^{j} b_{k-i}$ converges to p_j for $j \in \{0, \dots, k-1\}$. We also have that y_n converges to z. We conclude that (q_n, y_n) converges to (p, z), which must therefore be in the closure of Ω_0. □

Lemma 2.6. *The map $\psi : \Omega \to \mathbb{Z}^{A_k}$ is injective.*

Proof. Let $(p, z) \in \Omega'$ and define $f(c) := \psi_{p,z}(\Delta^{-c})$ for all $c \in \mathbb{Z}$. Since $\phi_i(\Delta^{-c}, z) = c$ for all $0 \leq i \leq k-1$, we have

$$f(c) = \sum_{i=0}^{k-1} |p_i + c| - \sum_{i=0}^{k-1} |p_i|.$$

Observe that, for $x \in \mathbb{N}$,

$$|x + c + 1| - |x + c| = \begin{cases} 1, & \text{if } x \geq -c, \\ -1, & \text{otherwise.} \end{cases} \tag{5}$$

So

$$\begin{aligned} f(c+1) - f(c) &= \sharp\{i \mid p_i \geq -c\} - \sharp\{i \mid p_i < -c\} \\ &= 2\sharp\{i \mid p_i \geq -c\} - k. \end{aligned}$$

Here $\sharp$ denotes the cardinal number of a set. Therefore, by calculating $\psi_{p,z}(\Delta^{-c-1}) - \psi_{p,z}(\Delta^{-c})$ for each $c \in \mathbb{N}$, we may determine the number of components of p that equal each element of $\mathbb{Z} \cup \{-\infty, +\infty\}$. Since the components of p are non-decreasing, we will then have determined p. Thus we have shown that if (p_1, z_1) and (p_2, z_2) are elements of Ω' such that $p_1 \neq p_2$, then $\psi_{p_1,z_1} \neq \psi_{p_2,z_2}$.

Now assume that $p_1 = p_2 =: p$ but that (p, z_1) and (p, z_2) are elements of distinct equivalence classes in Ω. So, p cannot be identically $+\infty$ or identically $-\infty$. We know from Lemma 2.4 that ψ is a bijection between Ω_0 and $\mathcal{D}$, so we may assume that not all entries of p are finite and that z_1 is an infinite word. Let x_n be the nth canonical factor of z_1 and let w_n be the product $w_n := x_1 \cdots x_n$.

We deal first with the case where p_0 is finite. For each canonical factor $y \in M^+$, denote by $l(y)$ the length of y, that is the total number of copies of a and b one has to multiply together to get y. Observe that $\phi(w_n, z) - \phi(w_{n-1}, z) = \phi(x_n, w_{n-1}^{-1} z)$ for any $z \in Z$. Since the effect of left multiplying $w_{n-1}^{-1} z_1$ by x_n^{-1} is to cancel exactly

one canonical factor of length $l(x_n)$, we get

$$\phi_i(w_n, z_1) - \phi_i(w_{n-1}, z_1) = \begin{cases} 0, & \text{if } i < k - l(x_n), \\ -1, & \text{otherwise}, \end{cases} \tag{6}$$

for all $i \in \{0, \dots, k-1\}$ and $n \in \mathbb{N}$. From (5), we see that

$$\begin{aligned} \psi_{p,z_1}(w_n) - \psi_{p,z_1}(w_{n-1}) &= \sum_{i=0}^{k-1} |p_i + \phi_i(w_n, z_1)| - \sum_{i=0}^{k-1} |p_i + \phi_i(w_{n-1}, z_1)| \\ &= -\sharp\{i \geq k - l(x_n) \mid p_i \geq -\phi_i(w_n, z_1)\} \\ &\qquad + \sharp\{i \geq k - l(x_n) \mid p_i < -\phi_i(w_n, z_1)\}. \end{aligned}$$

Since we have assumed that p_0 is finite and not all components of p are finite, we must have that $p_{k-1} = +\infty$. Therefore, the first set above is not empty, and so

$$\psi_{p,z_1}(w_n) - \psi_{p,z_1}(w_{n-1}) \leq l(x_n) - 2, \qquad \text{for all } n \in \mathbb{N}. \tag{7}$$

Now consider z_2. Since $z_2 \neq z_1$, eventually some x_n^{-1} will not cancel completely with the first canonical factor of $w_{n-1}^{-1} z_2$ and subsequent left multiplications by $x_{n+1}^{-1}, x_{n+2}^{-1}, \dots$ will have the effect of adding more factors. For each $n \in \mathbb{N}$, let y_n be such that $\Delta^{-1} y_n = x_n^{-1}$. Since y_n is a positive canonical factor of length $k - l(x_n)$, we get

$$\phi_i(w_n, z_2) - \phi_i(w_{n-1}, z_2) = \begin{cases} -1, & \text{if } i < l(x_n), \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

for all $i \in \{0, \dots, k-1\}$ and n large enough. So, for such n,

$$\begin{aligned} \psi_{p,z_2}(w_n) - \psi_{p,z_2}(w_{n-1}) &= \sum_{i=0}^{k-1} |p_i + \phi_i(w_n, z_2)| - \sum_{i=0}^{k-1} |p_i + \phi_i(w_{n-1}, z_2)| \\ &= -\sharp\{i < l(x_n) \mid p_i \geq -\phi_i(w_n, z_2)\} \\ &\qquad + \sharp\{i < l(x_n) \mid p_i < -\phi_i(w_n, z_2)\}. \end{aligned}$$

Let $i \in \{0, \dots, k-1\}$. If there are infinitely many $n \in \mathbb{N}$ such that $i < l(x_n)$, then, by (8), the sequence $\phi_i(w_n, z_2)$ is non-increasing and has limit $-\infty$. But our assumption on p implies that none of the p_i are equal to $-\infty$. Therefore, there are only a finite number of $n \in \mathbb{N}$ such that the first set above contains i. Since this is true for any i, the first set must eventually be empty.

So there are only finitely many n for which $\psi_{p,z_2}(w_n) - \psi_{p,z_2}(w_{n-1}) < l(x_n)$. Comparing this with (7), we see that ψ_{p,z_1} and ψ_{p,z_2} cannot be equal.

Now suppose that $p_0 = -\infty$. Note that $\phi_i(w\Delta^{-c}, z) = c + \phi_i(w, z)$ for all $w \in A_k$ and $0 \leq i \leq k-1$. So, using (6) and (8), we get

$$\phi_i(w_n \Delta^{-n}, z_1) - \phi_i(w_{n-1} \Delta^{-n+1}, z_1) = \begin{cases} 1, & \text{if } i < k - l(x_n), \\ 0, & \text{otherwise}, \end{cases}$$

for all $n \in \mathbb{N}$, and

$$\phi_i(w_n \Delta^{-n}, z_2) - \phi_i(w_{n-1} \Delta^{-n+1}, z_2) = \begin{cases} 0, & \text{if } i < l(x_n), \\ 1, & \text{otherwise}, \end{cases}$$

for n large enough. Using similar logic to that of the preceding case, we can show that

$$\psi_{p,z_1}(w_n\Delta^{-n}) - \psi_{p,z_1}(w_{n-1}\Delta^{-n+1}) \le l(x_n) - 2$$

for all $n \in \mathbb{N}$, and that

$$\psi_{p,z_2}(w_n\Delta^{-n}) - \psi_{p,z_2}(w_{n-1}\Delta^{-n+1}) = l(x_n)$$

for n large enough. So in this case also, ψ_{p,z_1} is different from ψ_{p,z_2}. □

Lemma 2.7. *The map $\psi : \Omega \to \mathbb{Z}^{A_k}$ is continuous.*

Proof. Let $((p^{(n)}, z^{(n)}))_{n\in\mathbb{N}}$ be a sequence in Ω converging to some element (p, z) of the same set in the topology we have chosen on Ω. If (p, z) is in Ω_0, then it is isolated and $(p^{(n)}, z^{(n)})$ must eventually be equal to it. So in this case, $\psi_{p^{(n)},z^{(n)}}$ obviously converges to $\psi_{p,z}$.

Now suppose that $p = (\infty, \dots, \infty)$. Observe that, for $w \in A_k$ fixed, $\phi(w, z^{(n)})$ is bounded uniformly in n. So, since each component of $p^{(n)}$ converges to ∞, we have, for each $w \in A_k$, that

$$\psi_{p^{(n)},z^{(n)}}(w) = \sum_{i=0}^{k-1} \phi_i(w, z^{(n)}), \qquad \text{for } n \text{ large enough.}$$

But, by Lemma 2.3, the right-hand-side is equal to $\sum_{i=0}^{k-1} \pi_i(w^{-1})$, and this is exactly $\psi_{+\infty}(w)$.

Similar reasoning shows that $\psi_{p^{(n)},z^{(n)}}$ converges to $\psi_{-\infty}$ if $p^{(n)}$ converges to $(-\infty, \dots, -\infty)$.

Suppose finally that (p, z) is in $\Omega\backslash\Omega_0$ and p is not identically either $+\infty$ or $-\infty$. Then $z^{(n)}$ converges to z and so, by Lemma 2.2, $\phi(w, z^{(n)})$ converges to $\phi(w, z)$ for each $w \in A_k$. Since also $p^{(n)}$ converges to p, we get that $\psi_{p^{(n)},z^{(n)}}$ converges to $\psi_{p,z}$ by inspecting the definition of ψ. □

Theorem 2.8. *The map ψ is a homeomorphism between Ω and $\mathcal{M}$.*

Proof. The injectivity of ψ was proved in Lemma 2.6 and so ψ is a bijection from Ω to $\psi(\Omega)$. As a continuous bijection from a compact space to a Hausdorff one, ψ must be a homeomorphism from Ω to $\psi(\Omega)$. So $\psi(\Omega)$ is compact and therefore closed. Since $\Omega = \operatorname{cl}\Omega_0$ by Lemma 2.5 and ψ is continuous by Lemma 2.7, we have $\psi(\Omega_0) \subset \psi(\Omega) \subset \operatorname{cl}\psi(\Omega_0)$. Taking closures, we get $\psi(\Omega) = \operatorname{cl}\psi(\Omega_0) = \mathcal{M}$, by Lemma 2.4. □

The proof of our characterisation of Busemann points will require a result from [1]: The Busemann points are precisely those horofunctions ξ for which $H(\xi, \xi) = 0$, where the *detour cost* $H(\cdot, \cdot)$ is defined by

$$H(\xi, \eta) := \liminf_{x\to\xi} \big(d(b, x) + \eta(x)\big)$$

for any pair of horofunctions ξ and η.

Theorem 2.9. *A function in $\mathcal{M}$ is a Busemann point if and only if the corresponding element (p, z) of Ω is in $\Omega\backslash\Omega_0$ and satisfies the following: $p_i - p_{i-1} = m_{k-i}(z)$ for every $i \in \{1, \dots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$.*

Proof. Assume $\xi \in \mathcal{M}$ is a Busemann point. So ξ is the limit of a sequence of group elements $x_n := y_0 \cdots y_n$, where y is an infinite geodesic word. Write $x_{n-1} = \Delta^r z_1 \cdots z_s$ in left normal form and let j be the length of the last canonical factor z_s. Consider the effect of right multiplying by y_n. There are four cases, corresponding to the four elements of S:

i. y_n is positive and $z_s y_n \in M^+ \cup \{\Delta\}$. In this case the length of the last canonical factor increases by one and so $\pi_{k-j-1}(x_n) = \pi_{k-j-1}(x_{n-1}) + 1$. All other components of $\pi(x_n)$ equal those of $\pi(x_{n-1})$;
ii. y_n is positive and $z_s y_n \notin M^+ \cup \{\Delta\}$. In this case another canonical factor y_n of length one is tacked onto the end and so $\pi_{k-1}(x_n) = \pi_{k-1}(x_{n-1}) + 1$, all other components being the same;
iii. y_n is negative and $z_s y_n \in M^+ \cup \{e\}$. In this case the length of the last canonical factor decreases by one and so $\pi_{k-j}(x_n) = \pi_{k-j}(x_{n-1}) - 1$, all other components being the same;
iv. y_n is negative and $z_s y_n \notin M^+ \cup \{e\}$. In this case we can see what happens more clearly by right multiplying x_n by $\Delta^{-1}(\Delta y_n)$ instead of y_n. Moving the Δ^{-1} all the way to the left, we see that the power of Δ becomes $r - 1$, each canonical factor z_i; $1 \le i \le s$ is replaced by $\tau(z_i)$, and another canonical factor Δy_n of length $k-1$ is tacked onto the end. So $\pi_0(x_n) = \pi_0(x_{n-1}) - 1$ and all other components stay the same.

In all cases, when going from $\pi(x_{n-1})$ to $\pi(x_n)$, a single component is changed, either increased of decreased by one. Looking at the distance formula of Proposition 2.1, we see that, since y is a geodesic word, an increase is only possible when the relevant component of $\pi(x_{n-1})$ is non-negative, and a decrease is only possible when it is non-positive.

If case (i) occurs infinitely often with $j = k-1$, then $\pi_0(x_n)$ converges to $+\infty$ as n tends to infinity, and so every component of $\pi(x_n)$ converges to $+\infty$. In this case, the condition in the statement of the theorem holds trivially. So we may assume that case (i) occurs only finitely many times with $j = k-1$. Likewise, we may assume that case (iii) occurs only finitely many times with $j = 1$.

One sees that case (ii) creates a new canonical factor of length one, which can be lengthened by successive applications of case (i), whereas case (iv) creates a new canonical factor of length $k-1$, which can be shortened by successive applications of case (iii). For each $n \in \mathbb{N}$, denote by $z^{(n)}$ the word consisting of all the canonical factors of x_n taken in sequence. Because of the assumptions of the previous paragraph, eventually, once a canonical factor has been created it can not be removed. So if we take the sequence of times $(n_t)_{t\in\mathbb{N}}$ where either case (ii) or case (iv) occurs, then the difference between $z^{(n_t)}$ and $z^{(n_{t-1})}$ is that a new canonical factor has been added and, possibly, that the original canonical factors have been operated on by τ.

Fix $i \in \{1, \ldots, k-1\}$ such that p_{i-1} and p_i are not both $+\infty$ nor both $-\infty$. We have that $\pi_i(x_{n_t}) - \pi_{i-1}(x_{n_t})$ is equal to $m_{k-i}^{n_t}$, the number of canonical factors of length $k-i$ in $z^{(n_t)}$. But because $z^{(n_t)}$ grows monotonically as t increases, $m_{k-i}^{n_t}$ converges as t tends to infinity to $m_{k-i}(z)$, the number of canonical factors of length $k-i$ in z. Therefore,

$$p_i - p_{i-1} = \lim_{t\to\infty} (\pi_i(x_{n_t}) - \pi_{i-1}(x_{n_t})) = \lim_{t\to\infty} m_{k-i}^{n_t} = m_{k-i}(z).$$

This establishes the implication in one direction.

Now assume that $\xi \in \mathcal{M}$ corresponds to $+\hat{\infty}$. For each $n \in \mathbb{N}$, let $x_n := \mathrm{prod}(a, b; n)$ and let $(p^{(n)}, z^{(n)})$ be the corresponding element of Ω_0. We see that $p_0^{(n)} = \lfloor n/k \rfloor$, which tends to infinity as n tends to infinity. It follows that $p^{(n)}$ converges to $(+\infty, \ldots, +\infty)$ and hence x_n converges to ξ by Theorem 2.8. Since x_n is a geodesic, ξ must be a Busemann point.

When $\xi \in \mathcal{M}$ corresponds to $-\hat{\infty}$, we take $x_n := \mathrm{prod}(a^{-1}, b^{-1}; n)$ and use a similar argument.

Now assume that ξ corresponds to an element $(p, z) \in \Omega \backslash (\Omega_0 \cup \{-\hat{\infty}, +\hat{\infty}\})$ satisfying the condition in the statement of the theorem. Let w^n be the word consisting of the first n canonical factors of z. Let $j \in \{0, \ldots, k-1\}$ be the index of either the first non-negative component of p or the last non-positive component. We can choose a sequence of vectors q^n in $\mathbb{Z}^k$ such that $q_i^n - q_{i-1}^n = m_{k-i}(w^n)$ for all $i \in \{1, \ldots, k-1\}$, and such that q_j^n converges to p_j. Since $(q^n, w^n) \in \Omega_0$ for all $n \in \mathbb{N}$, we may consider the element x^n of A_k corresponding to (q^n, w^n). From our assumption on ξ, we have that $m_{k-i}(w^n)$ converges as n tends to infinity to $p_i - p_{i-1}$ for all $i \in \{1, \ldots, k-1\}$ such that p_i and p_{i-1} are not both $+\infty$ nor both $-\infty$.

Using this and the definition of q^n, we conclude that q^n converges to p as n tends to infinity. But we also have that w^n converges to z and so, by Theorem 2.8, x^n converges to ξ. Multiplying z on the left by $(x^n)^{-1}$ has the effect of canceling $m_i(w^n)$ factors of length i for each $i \in \{1, \ldots, k-1\}$ and adding a factor $\Delta^{-q_0^n}$. Therefore

$$\phi_i(x^n, z) = -q_0^n - m_{k-1}(w^n) - \cdots - m_{k-i}(w^n) = -q_i^n.$$

So

$$\begin{aligned} H(\xi, \xi) &\leq \liminf_{n\to\infty} (d(e, x^n) + \psi_{p,z}(x^n)) \\ &= \liminf_{n\to\infty} \sum_{i=0}^{k-1} \Big(|q_i^n| + |p_i - q_i^n| - |p_i| \Big) \\ &= 0, \end{aligned}$$

since q^n converges to p. This proves that ξ is a Busemann point. □

3. Dual generators

We establish a formula for the dual-generator word-metric using a technique originally developed by Fordham [10] to prove a length formula for Thompson's group F. The following theorem is a right-handed version of one in [2].

Theorem 3.1. *Let G be a group with generating set S, and let $l : G \to \mathbb{N}$ be a function. Then l gives the distance with respect to S from the identity to any given element if and only if*

L1. $l(e) = 0$,

L2. $|l(wg) - l(w)| \leq 1$ *for all* $w \in G$ *and* $g \in S$,

L3. *if* $w \in G \backslash \{e\}$, *then there exists* $g \in S \cup S^{-1}$ *such that* $l(wg) < l(w)$.

Proposition 3.2. *Let $w = \delta^r w_1 w_2 \cdots w_s$ be written in left normal form with respect to the dual generators. Then the distance between the identity and w with respect to these generators is given by $\tilde{d}(e, w) = |r| + |r + s|$.*

Proof. Let $l(w) := |r| + |r+s|$. Clearly l satisfies (L1). Consider the effect of right multiplying w by a generator $g \in \tilde{S}$. Let $v := wg$ and write this group element in left normal form $v = \delta^{r'} v_1 v_2 \cdots v_{s'}$. There are four cases to consider:

i. g is positive and $w_s g = \delta$. In this case $r' = r + 1$ and $s' = s - 1$.
ii. g is positive and $w_s g \neq \delta$. In this case $r' = r$ and $s' = s + 1$.
iii. g is negative and $w_s g = e$. In this case $r' = r$ and $s' = s - 1$.
iv. g is negative and $w_s g \neq e$. In this case $r' = r - 1$ and $s' = s + 1$.

In all cases, either $r' = r$ and $r' + s' = r + s \pm 1$, or $r' = r \pm 1$ and $r' + s' = r + s$. Therefore (L2) is satisfied.

Also, by choosing g appropriately, we can make whichever of the four cases we want happen. So we always have the freedom to increase or decrease either r or $r + s$ by one. It follows that (L3) holds. ☐

We note that an algorithm for finding a geodesic representative of any given word in A_3 with respect to the dual generators was presented in [18].

Observe that the distance formula above has a form similar to the formula established in Proposition 2.1 for the distance with respect to the Artin generators. This similarity will allow us to calculate the horofunction boundary and the Busemann points with respect to the dual generators using the same method as for the Artin generators.

As before we define some maps. For any $w \in A_k$, let $\tilde{m}_1(w)$ and $\tilde{m}_2(w)$ be such that w can be written in left normal form as $w = \delta^{\tilde{m}_2(w)} w_1 \cdots w_{\tilde{m}_1(w)}$. Define $\tilde{\pi} : A_k \to \mathbb{Z}^2$ by

$$\tilde{\pi}(w) := (\tilde{m}_2(w), \tilde{m}_1(w) + \tilde{m}_2(w)).$$

Finally, let

$$\tilde{\phi}(w, z) := \tilde{\pi}(w^{-1}z) - \tilde{\pi}(z), \qquad \text{for all } w \text{ and } z \text{ in } A_k.$$

The proof of the following lemma is similar to its counterpart, Lemma 2.2.

Lemma 3.3. *Let $w \in A_k$ and let $z_1 z_2 \cdots$ be an infinite word of positive dual generators such that no product of consecutive letters equals δ. Then $\tilde{\phi}(w, z_1 \cdots z_n)$ converges as n tends to infinity.*

Let $\tilde{Z}$ be the set of possibly infinite words of positive dual generators having no product of consecutive letters equal to δ. The previous lemma allows us to define $\tilde{\phi}(w, z)$ for $w \in A_k$ and $z = z_1 z_2 \cdots$ an infinite element of $\tilde{Z}$ to be the limit of $\tilde{\phi}(w, z_1 \cdots z_n)$ as n tends to infinity.

Let $\tilde{\Omega}'$ denote the set of (p, z) in $(\mathbb{Z} \cup \{-\infty, +\infty\})^2 \times \tilde{Z}$ such that if p is not identically $-\infty$ nor identically $+\infty$, then $p_1 - p_0 = \tilde{m}_1(z)$. We take the product topology on $\tilde{\Omega}'$. Let $\tilde{\Omega}$ be the quotient of $\tilde{\Omega}'$ obtained by considering all points $((-\infty, -\infty), z)$ with $z \in \tilde{Z}$ to be equivalent, and all points $((+\infty, +\infty), z)$ with $z \in \tilde{Z}$ to be equivalent. The former equivalence class we denote simply by $-\tilde{\infty}$, the latter by $+\tilde{\infty}$.

For each $(p, z) \in \tilde{\Omega}'$, define

$$\tilde{\psi}_{p,z} : A_k \to \mathbb{Z}, \quad w \mapsto |p_0 + \tilde{\phi}_0(w, z)| + |p_1 + \tilde{\phi}_1(w, z)| - |p_0| - |p_1|. \tag{9}$$

We use the same convention as before for adding and subtracting infinities. The following lemma shows that $\tilde{\psi}$ is constant on the equivalence classes $-\tilde{\infty}$ and $+\tilde{\infty}$. The proof of this lemma is the same as that of Lemma 2.3.

Lemma 3.4. *For all w and z in A_k,*

$$\tilde{\phi}_0(w,z) + \tilde{\phi}_1(w,z) = \tilde{\pi}_0(w^{-1}) + \tilde{\pi}_1(w^{-1}).$$

So we see that if $p = (-\infty, -\infty)$, then

$$\tilde{\psi}_{p,z}(w) = -\tilde{\pi}_0(w^{-1}) - \tilde{\pi}_1(w^{-1})$$

is independent of z. Likewise, if $p = (+\infty, +\infty)$, then

$$\tilde{\psi}_{p,z}(w) = \tilde{\pi}_0(w^{-1}) + \tilde{\pi}_1(w^{-1}).$$

We may therefore consider the map $\tilde{\psi}$ to be defined on $\tilde{\Omega}$.

Let $\tilde{\mathcal{D}} := \{\tilde{d}(\cdot, x) - \tilde{d}(e, x) \mid x \in A_k\}$ and let $\tilde{\mathscr{M}}$ be its closure, that is, the horofunction compactification of A_k with the dual-generator word metric.

Let $\tilde{Z}_0$ be the set of finite words with letters in $\{\sigma_1, \ldots, \sigma_k\}$ having no product of consecutive letters equal to δ and define

$$\tilde{\Omega}_0 := \{(p, z) \in \mathbb{Z}^2 \times \tilde{Z}_0 \mid p_1 - p_0 = \tilde{m}_1(z)\}.$$

Again, we wish to show that $\tilde{\Omega}$ is homeomorphic to $\tilde{\mathscr{M}}$ with $\tilde{\Omega}_0$ being mapped to $\tilde{\mathcal{D}}$. We use the same method we used for the Artin generators. The proofs of the following results are similar to those of the corresponding results in Section 2.

Lemma 3.5. *Restricted to $\tilde{\Omega}_0$, the map $\tilde{\psi}$ is a bijection between $\tilde{\Omega}_0$ and $\tilde{\mathcal{D}}$.*

Lemma 3.6. *The set $\tilde{\Omega}_0$ is dense in $\tilde{\Omega}$.*

Lemma 3.7. *The map $\tilde{\psi} : \tilde{\Omega} \to \mathbb{Z}^{A_k}$ is injective.*

Lemma 3.8. *The map $\tilde{\psi} : \tilde{\Omega} \to \mathbb{Z}^{A_k}$ is continuous.*

Theorem 3.9. *The map $\tilde{\psi}$ is a homeomorphism between $\tilde{\Omega}$ and $\tilde{\mathscr{M}}$.*

The proof of the following theorem uses the same reasoning as that of Theorem 2.9.

Theorem 3.10. *In the horoboundary of A_k with the dual-generator word metric, all horofunctions are Busemann points.*

We use our distance formula to characterise the geodesic words of A_k with the dual generators.

Proposition 3.11. *Let $x \in A_k$ and let y be a freely reduced word of dual generators representing x. Then y is a geodesic if and only if $\widetilde{\mathrm{Pos}}(y) + \widetilde{\mathrm{Neg}}(y) \le 2$.*

Proof. Let y be such that $\widetilde{\mathrm{Pos}}(y) + \widetilde{\mathrm{Neg}}(y) > 2$. Since neither $\widetilde{\mathrm{Pos}}(y)$ nor $\widetilde{\mathrm{Neg}}(y)$ are greater than 2, one of them must equal 2 and the other must be positive. Suppose $\widetilde{\mathrm{Pos}}(y) = 2$ and $\widetilde{\mathrm{Neg}}(y) > 0$. Then y contains a negative generator and two consecutive positive generators with product δ. Take the δ and shift it towards the negative generator by repeatedly using the relations $\sigma_i \delta = \delta \sigma_{i+2}$ and $\sigma_i^{-1}\delta = \delta\sigma_{i+2}^{-1}$. Then cancel the negative generator with the δ using $\sigma_i^{-1}\delta = \sigma_{i+1}$. The result is a word representing x that is shorter by one generator than y. Therefore y is not a geodesic. The proof in the case when $\widetilde{\mathrm{Pos}}(y) > 0$ and $\widetilde{\mathrm{Neg}}(y) = 2$ is similar.

Now assume that $\widetilde{\mathrm{Neg}}(y) = 0$. Consider what happens if we start at the identity and successively multiply by generators as prescribed by y. We obtain a sequence, which we denote by $(x_n)_{n\in\mathbb{N}}$. Initially $r = r + s = 0$, where r and s are as in

Proposition 3.2. Since y is composed only of positive generators, only cases (i) and (ii) in the proof of Proposition 3.2 are relevant here. We note that in these two cases, either r or $r+s$ increases by one, and the other stays the same. Therefore $\tilde{d}(e, x_n) = n$. It follows that y is a geodesic.

The proof that y is a geodesic if $\widetilde{\mathrm{Pos}}(y) = 0$ is similar. The cases concerned this time are (iii) and (iv), and in both of these either r or $r+s$ decreases by one and the other stays the same.

The final case to consider is when $\widetilde{\mathrm{Pos}}(y) = \widetilde{\mathrm{Neg}}(y) = 1$. We claim that, as the generators comprising y are successively multiplied, the rightmost canonical factor in the left normal form of x_n is equal to y_n when y_n is positive and equal to δy_n when y_n is negative. To show this, we use induction on n. Suppose the claim is true for x_n, which we write in left normal form as $x_n = \delta^r w_1 \cdots w_s$. If y_n is positive, our induction hypothesis gives that $w_s = y_n$, and so y_{n+1} can not equal either w_s^{-1} or $w_s^{-1}\delta$ since y is freely reduced and $\widetilde{\mathrm{Pos}}(y) < 2$. Therefore, if y_n is positive, neither case (i) nor case (iii) of Proposition 3.2 can occur. Since there is no cancellation, the left normal form of x_{n+1} has then y_{n+1} or δy_{n+1} as rightmost canonical factor, depending on whether y_{n+1} is positive or negative. Similar reasoning shows the same is true when y_n is negative. Thus we have proved our claim.

The argument of the previous paragraph also established that cases (i) and (iii) of Proposition 3.2 never occur when x_n is multiplied on the right by y_{n+1}.

In case (ii) of that proposition, $r+s$ increases by one while r remains the same, and in case (iv), r decreases by one while $r+s$ remains the same. Therefore, $|r| + |r+s|$ always increases by one as each letter of y is added, and so $\tilde{d}(e, x_n) = n$. So in this case also, y is a geodesic. $\square$

This characterisation of geodesics allows us to calculate the geodesic growth series of A_k.

Theorem 3.12. *The geodesic growth series of A_k with the dual generators is*

$$\mathcal{G}(x) = \frac{1 + (3-2k)x + (2+k^2-3k)x^2 - 2k(k-1)x^3}{(1-kx)(1-2(k-1)x)(1-(k-1)x)}.$$

Proof. Let N_{ij}^n be the number of freely reduced words y of length n satisfying $\widetilde{\mathrm{Pos}}(y) \le i$ and $\widetilde{\mathrm{Neg}}(y) \le j$, and let $\mathcal{G}_{ij}$ be the corresponding generating series. Proposition 3.11 and an inclusion–exclusion argument give that the number of geodesics of length n is

$$N_{20}^n + N_{02}^n + N_{11}^n - N_{10}^n - N_{01}^n.$$

Therefore

$$\mathcal{G} = \mathcal{G}_{20} + \mathcal{G}_{02} + \mathcal{G}_{11} - \mathcal{G}_{10} - \mathcal{G}_{01}. \tag{10}$$

Clearly, $N_{20}^n = N_{02}^n = k^n$ for all $n \in \mathbb{N}$, and so

$$\mathcal{G}_{20}(x) = \mathcal{G}_{02}(x) = 1 + kx + k^2x^2 + \cdots = \frac{1}{1-kx}.$$

Consider now the freely reduced words not containing δ or δ^{-1} as sub-words. For the first letter we may choose any of the $2k$ generators. For subsequent letters, we can choose any letter apart from the inverse of the previous one and the letter

that would combine with the previous one to form δ or δ^{-1}. So we have a choice of $2k-2$ generators. Therefore the growth series $\mathcal{G}_{11}$ for this set of words is

$$\begin{aligned}\mathcal{G}_{11}(x) &= 1 + 2kx + 2k(2k-2)x^2 + 2k(2k-2)^2x^3 + \cdots \\ &= \frac{1+2x}{1-2(k-1)x}.\end{aligned}$$

Now consider the set of freely reduced words containing only positive generators and no sub-word equal to δ. This time there are k possibilities for the first letter and $k-1$ for subsequent letters. So the growth series is

$$\begin{aligned}\mathcal{G}_{10}(x) &= 1 + kx + k(k-1)x^2 + k(k-1)^2x^3 + \cdots \\ &= \frac{1+x}{1-(k-1)x}.\end{aligned}$$

The growth series $\mathcal{G}_{01}$ is identical.

The conclusion now follows from (10) after some rearranging. □

The first few terms of $\mathcal{G}(x)$ are

$$\mathcal{G}_(x) = 1 + 2kx + 2(2k^2 - k)x^2 + 2(k^3 + 3k(k-1)^2)x^3 + \cdots .$$

References

[1] Marianne Akian, Stéphane Gaubert, and Cormac Walsh. The max-plus Martin boundary. Preprint. arXiv:math.MG/0412408, 2004.

[2] James M. Belk and Kenneth S. Brown. Forest diagrams for elements of Thompson's group F. *Internat. J. Algebra Comput.*, 15(5-6):815–850, 2005.

[3] Mitchell A. Berger. Minimum crossing numbers for 3-braids. *J. Phys. A*, 27(18):6205–6213, 1994.

[4] David Bessis. The dual braid monoid. *Ann. Sci. École Norm. Sup. (4)*, 36(5):647–683, 2003.

[5] Ruth Charney and John Meier. The language of geodesics for Garside groups. *Math. Z.*, 248(3):495–509, 2004.

[6] A. Connes. Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems*, 9(2):207–220, 1989.

[7] Michel Coornaert and Athanase Papadopoulos. Horofunctions and symbolic dynamics on Gromov hyperbolic groups. *Glasg. Math. J.*, 43(3):425–456, 2001.

[8] Mike Develin. Cayley compactifications of abelian groups. *Ann. Comb.*, 6(3-4):295–312, 2002.

[9] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

[10] S. Blake Fordham. Minimal length elements of Thompson's group F. *Geom. Dedicata*, 99:179–220, 2003.

[11] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213, Princeton, N.J., 1981. Princeton Univ. Press.

[12] Jean Mairesse and Frédéric Mathéus. Growth series for Artin groups of dihedral type. Preprint, 2005.

[13] Marc A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651 (electronic), 2002.

[14] Lucas Sabalka. Geodesics in the braid group on three strands. In *Group theory, statistics, and cryptography*, volume 360 of *Contemp. Math.*, pages 133–150. Amer. Math. Soc., Providence, RI, 2004.

[15] Peter A. Storm. The barycenter method on singular spaces. Preprint. arXiv:math.GT/0301087, 2003.

[16] Corran Webster and Adam Winchester. Boundaries of hyperbolic metric spaces. *Pacific J. Math.*, 221(1):147–158, 2005.

[17] Corran Webster and Adam Winchester. Busemann points of infinite graphs. *Trans. Amer. Math. Soc.*, 358(9):4209–4224 (electronic), 2006.
[18] Peijun Xu. The genus of closed 3-braids. *J. Knot Theory Ramifications*, 1(3):303–326, 1992.

INRIA, Domaine de Voluceau, 78153 Le Chesnay Cédex, France
E-mail address: `cormac.walsh@inria.fr`